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PLANE INVOLUTIONS OF ORDER FOUR.

By Temple Rice Hollcroft.

1. A plane involution of order four or a (1, 4) point correspondence between two planes is said to exist when to one point in the first plane correspond four points in the second to each of which corresponds the original point in the first. The planes may be distinct or coincident.

Plane involutions of order two have been completely classified by Bertini[*] and those of order three by Miss Howe.[†] Miss Scott[‡] has treated metrically a case of (1, 4) point correspondence which is a special form of type 3 of this classification.

The purpose of this paper is to discuss and classify the algebraic methods of relating two planes in (1, 4) point correspondence. All the methods that are reducible to each other by means of birational transformations will be counted as one, and the most general of these will be called a type of (1, 4) point correspondence. The types are thus birationally independent.

The general methods employed are due to Sharpe and Snyder[§] and have been made use of since by A. M. Howe.[||] The new features of plane involutions of order four over those of order three and two are found principally in the discussion of coincident images. Involutions of order higher than four, however, present no new features, that is, an involution of order four may be generalized for any order n as demonstrated in the latter part of this paper. It is felt, therefore, that involutions of order four are of sufficient importance to warrant their classification, since they are of the highest order that has distinctive characteristics and those of order n cannot be classified.

2. A type is defined by two algebraic equations in x_1, x_2, x_3 and x'_1, x'_2, x'_3 of the form

$$x'_1u_1 + x'_2u_2 + x'_3u_3 = 0, \tag{1}$$

$$x'_1v_1 + x'_2v_2 + x'_3v_3 = 0, \tag{2}$$

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[*] E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244–286.

[†] A. M. Howe, "A Classification of Plane Involutions of Order Three," American Journal of Math., Vol. XLI (1919), pp. 25–40.

[‡] C. A. Scott, "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. 29 (1899), pp. 329–381, and Vol. 32 (1901), pp. 209–239.

[§] F. R. Sharpe and V. Snyder, "Types of (2, 2) point correspondences between two planes," *T. A. M. S.*, Vol. 18 (1917), pp. 402–414.

[||] A. M. Howe, loc. cit.

wherein $u_i = 0$ and $v_i = 0$ are curves of (x) intersecting in four variable points. These equations so relate the planes (x) and (x') that to a point P' of (x') correspond P_1, P_2, P_3, P_4 of (x) and to each of these image points of (x) corresponds the original point P', for the coördinates of a point of (x') determine uniquely a pair of curves of (x) intersecting in four non-basic points and the two curves so determined are in (1, 1) correspondence with the coördinates of the point of (x'). Ordinarily the four image points will be distinct, but the coördinates of P' may be so chosen that the corresponding curves of (x) shall have one or two contacts. The locus of such points P' is a curve of (x') called the curve of branch points. It will be denoted by the letter L'. The locus of the contacts in (x) is called the coincidence curve, denoted by K. When two image points coincide on K, the other two lie on a residual curve Γ. The complete image of K is L'; of L', $K^2\Gamma$; of Γ, L'^2. L' and K are in (1, 1) correspondence; L' and Γ (and therefore K and Γ) are in (1, 2) point correspondence.

To a line of (x') corresponds a curve of (x) whose image is the original line of (x') counted four times. To two lines of (x') intersecting at P' correspond two curves of (x) intersecting in four non-basic points, the images of P'. The remaining intersections of the two image curves of (x) are fixed points common to all image curves of (x). Since these curves are in (1, 1) correspondence with the lines of (x') they form a net. The basis points of this net are the fixed points common to all the image curves. Their images in (x') are basis curves the order of the curve being equal to the multiplicity of the point on the line images. When a given curve of (x) passes through a basis point, its image in (x') is composite, consisting of the curve-image of the basis point (counted as many times as the multiplicity of the basis point on the given curve) and a curve called the proper image of the given curve of (x).

The jacobian of the net of line image curves of (x) is the locus of the contacts of the curves of the net. It therefore either is the coincidence curve K or contains it as a factor. In the latter case the other factors are basis curves of (x). The residual curve Γ is sometimes called the co-jacobian.

To the lines of (x) correspond rational curves of (x'). Since the transformation from (x') to (x) is not rational the image curves of (x') can have basis points only of higher multiplicity than two. The images of these basis points are irrational basis curves of (x) whose order is the multiplicity of the basis point on the line images of (x').

3. There are ten independent types of (1, 4) point correspondences. Each type is defined by two equations of the form described in the preceding section. The equations are both linear in (x'), so the types depend only

on the choice of u_i and v_i of (x). The following table shows the curves represented by u_i and v_i for each type. Subscripts of C denote the order of a curve, coefficients and subscripts of P the number and multiplicity respectively of its basis points.

Type	$u_i = 0$	$v_i = 0$
1	line pencil	C_n; P_{n-4}
2	line	quartic
3	cubic; $5P_1$	cubic; $5P_1$
4	conic	conic
5	cubic; $8P_1$	C_{12}; $8P_4$
6	cubic; $8P_1$	quartic; $8P_1$
7	conic; P_1	cubic; P_2
8	conic; $2P_1$	cubic; $2P_1$
9	conic; $2P_1$	quartic; $2P_2$
10	cubic; $7P_1$	quartic; P_2, $6P_1$

4. *Type 1.*—The defining equations are

$$x_1x'_1 + x_2x'_2 = 0, \tag{1}$$

$$x'_1\varphi_1 + x'_2\varphi_2 + x'_3\varphi_3 = 0, \tag{2}$$

$\varphi_i(x) \equiv x_3^4u_i(x_1, x_2) + x_3^3v_i(x_1, x_2) + x_3^2w_i(x_1, x_2) + x_3s_i(x_1, x_2) + t_i(x_1, x_2) = 0,$ wherein u_i, v_i, w_i, s_i, t_i are homogeneous functions of the respective degrees $n - 4$, $n - 3$, $n - 2$, $n - 1$, n in x_1, x_2. Then equation (2) represents a C_n with an $(n - 4)$-fold point at $P \equiv (0, 0, 1)$, the vertex of the line pencil. The equations of transformation from (x') to (x) are

$$\begin{aligned} \rho x'_1 &= -\, x_2\varphi_3, \\ \rho x'_2 &= x_1\varphi_3, \\ \rho x'_3 &= x_2\varphi_1 - x_1\varphi_2. \end{aligned}$$

A line of (x') with coefficients a'_1, a'_2, a'_3 corresponds to

$$C_{n+1} \equiv (a'_1x_2 - a'_2x_1)\varphi_3 + a'_3(x_1\varphi_2 - x_2\varphi_1) = 0.$$

C_{n+1} has P_{n-3} and is of genus $3n - 6$. The curves $\varphi_3 = 0$ and $x_1\varphi_2 - x_2\varphi_1 = 0$ intersect in $8n - 12$ points outside of P. These $8n - 12$ P_1 are simple basis points of (x) through which pass all image curves of (x). Two line image curves intersect in four non-basic points which are collinear with P and are the images of the intersection of the two lines of (x'). If the line of (x') passes through $P' \equiv (0, 0, 1)$, its image is

$$\varphi_3(a'_1x_2 - a'_2x_1) = 0.$$

The curve $\varphi_3 = 0$ is fundamental, the image of P', and the other factor is the proper image of the line through P'.

To a line of (x) with coefficients a_1, a_2, a_3 corresponds

$$C'_{n+1} \equiv x'_1\varphi'_1 + x'_2\varphi'_2 + x'_3\varphi'_3 = 0,$$

wherein φ'_i has the same form as φ_i with $-a_3x'_2$, $a_3x'_1$ and $a_1x'_2 - a_2x'_1$ substituted respectively for x_1, x_2 and x_3. C'_{n+1} has an n-fold point at P' and is of genus 0. The proper image of a line through P is a line through P' counted four times. The basis point P has for its image the basis curve

$$u' \equiv x'_1u_1(-x'_2, x'_1) + x'_2u_2(-x'_2, x'_1) + x'_3u_3(-x'_2, x'_1) = 0.$$

The basis curve u' is of order $n - 3$ with P'_{n-4}.

Each of the $8n - 12$ P_1 corresponds to a line of (x') through P' and tangent to L'.

5. The equation of the branch-point curve L' is the condition on the parameters x'_1 that the line and C_n given by the defining equations have a point of contact. This condition is of degree $6n - 12$ in the coefficients of the line and 6 in the coefficients of the C_n. L' is therefore of order $6n - 6$ with P'_{6n-12}. L' will also be determined as the image of K.

The jacobian of the net of line images in (x) is

$$\varphi_3\left[\varphi_3\left(x_2\frac{\partial\varphi_1}{\partial x_3} - x_1\frac{\partial\varphi_2}{\partial x_3}\right) - \frac{\partial\varphi_3}{\partial x_3}(x_2\varphi_1 - x_1\varphi_2)\right] = 0.$$

The jacobian consists of the basis curve $\varphi_3 = 0$ and the coincidence curve K. K is of order $2n$ and genus $10n - 20$ with P_{2n-6} and $8n - 12$ P_1. The image of K is L'_{6n-6} with P'_{6n-12} and of genus $10n - 20$. Then L' has the equivalent of $20n - 30$ double points.

If in the two defining equations we solve the equations of the line and C_n simultaneously, simplify and arrange in powers of x_3/x_1, the discriminant of the quartic in x_3/x_1 is the equation of L'. Thus

$$L'_{6n-6} \equiv S^3 + 27T^2,$$

wherein

$$S \equiv 4s'v' - u't' - 3w'^2,$$
$$T \equiv u'w't' + 2v'w's' - v'^2t' - u's'^2 - w'^3;$$

u' is the fundamental curve of (x') and v', w', s', t' are curves whose equations are like that of u' with v, w, s, t substituted respectively for u. The curves S and T intersect in $12n - 18$ non-basic points, which are cusps of L'. Then of the $20n - 30$ double points of L' $12n - 18$ are cusps and $8n - 12$, nodes. The coördinates of a cusp of L' determine the line and C_n of the defining equations so that the line is tangent to the C_n at a point of inflection, and the coördinates of a double point of L' so that the line is a double tangent to the C_n.

The class of L' is $20n - 36$. Then from P' $8n - 12$ tangents can be drawn to L' which are the basis lines of (x').

The image of L' is a curve of order $12n - 6$ consisting of K counted twice and the residual curve Γ_{8n-6} which has P_{8n-18}, $8n - 12P_4$, $12n - 18$ cusps and is of genus $28n - 53$.

K and Γ have $40n - 60$ intersections among which are $12n - 18$ contacts, the common tangent passing through P. These contacts correspond to the cusps of L', that is, at these contacts three image points coincide. The fourth image point lies at a cusp of Γ on this common tangent. The remaining $16n - 24$ are simple intersections, images of the nodes of L'. To a node of L' correspond two intersections of K and Γ such that the line joining them is tangent to Γ at both points and passes through P. The two intersections represent a pair of coincidences.

The class of K is $24n - 42$. From P $20n - 30$ tangents can be drawn to K consisting of $8n - 12$ at the simple basis points and $12n - 18$ at the contacts of K and Γ. The class of Γ is $60n - 102$. From P $44n - 66$ tangents may be drawn to Γ which consist of $12n - 18$ at the contacts of K and Γ, $8n - 12$ bitangents through P and the $8n - 12$ P_1 and $8n - 12$ bitangents at the pairs of coincidences.

The four images of a point of (x') cannot ordinarily coincide because a general C_n of the defining equations has no point of undulation. When the equation of the C_n is so chosen that it will have such a point, certain of the pairs of coincidences will themselves coincide making four coincident images. Such a point is an undulation of Γ, the double tangent passing through P and a node of K neither of whose branches has contact with Γ. The corresponding point of L' is a cusp of the second kind.

6. L' and the basis curve u' intersect in $12n - 30$ points of which $2n - 6$ are contacts corresponding to the $2n - 6$ directions of K through P and $8n - 18$ intersections corresponding to the directions of Γ through P.

The image of the basis curve u' is P_{n-3} and a residual curve p of order $2n - 3$ with P_{2n-6} and $8n - 12$ P_1. K and p have the same tangents at P. K meets p in $10n - 24$ points besides at basis points. Of these $2n - 6$ lie on the $2n - 6$ common tangents to K and p at P and $8n - 18$ lie on the $8n - 18$ tangents to Γ at P. The class of p is $12n - 30$. From P $8n - 18$ tangents can be drawn to p. These tangents coincide with the tangents to Γ at P and pass through intersections of p with K and of p with Γ. Γ and p intersect in $16n - 42$ points of which $8n - 24$ are $4n - 12$ contacts lying two each on the $2n - 6$ common tangents to p and K at P and $8n - 18$ intersections lying on the tangents to p from P and to Γ at P. The complete image of p is u' counted three times.

The image of the point of contact of a basis line of (x') and L' consists of two consecutive points of K at the basis point and two pairs of consecutive points of Γ not at the basis point. Then the line joining P to a P_1 has a contact with K at P_1 and two contacts with Γ not at P_1. A line through P and a P_1 has for its image the basis line of (x') corresponding to that P_1 counted three times.

7. For the remaining types only the most important results are given. The defining equations for each type may be constructed from the table in section 3. The following notation will be used in tabulating results:

The symbol $\sim$ meaning "corresponds to."

L', K, Γ, curves as heretofore described.

C, C', variable curves of (x) and (x').

$\underline{P}$, $\underline{P}'$, basis points of (x) and (x').

$\overline{P}$, $\overline{P}'$, variable points of (x) and (x').

k, double points of L'.

Subscripts of curves denote their order.

Subscripts of points denote their multiplicity on the curve being described.

Type 2.
$C'_1 \sim C_5,\ 21\underline{P}_1.$
$C_1 \sim C'_5,\ \overline{P}'_4.$
$L'_{18},\ 102k \sim K^2_{12},\ 21P_2;\ \Gamma_{66},\ 21P_{14}.$
Type 3.
$C'_1 \sim C_6,\ 5\underline{P}_2,\ 12P_1.$
$C_1 \sim C'_6,\ 10\overline{P}'_2.$
$L'_{16},\ 76k \sim K^2_{15},\ 5P_5,\ 12P_2;\ \Gamma_{66},\ 5P_{22},\ 12P_{12}.$
Type 4.
$C'_1 \sim C_4,\ 12\underline{P}_1.$
$C_1 \sim C'_4,\ 3\overline{P}'_2.$
$L'_{12},\ 39k \sim K^2_9,\ 12P_2;\ \Gamma_{30},\ 12P_8.$
Type 5. Q_1 is the ninth basis point of the cubic pencil.
$C'_1 \sim C_{15},\ 8P_5,\ \underline{Q}_1,\ 20P_1.$
$C_1 \sim C'_{15},\ P'_{12},\ 25\overline{P}'_2.$
$Q_1 \sim$ line not through P'.
$P_1 \sim$ line through P'.
$L'_{28},\ P'_{20},\ 116k \sim K^2_{30},\ 8P_{10},\ Q_2,\ 20P_1;\ \Gamma_{120},\ 8P_{40},\ Q_{24},\ 20P_6.$
Type 6.
$C'_1 \sim C_7,\ 8P_2,\ \underline{Q}_1,\ 12P_1.$
$C_1 \sim C'_7,\ P'_4,\ 9\overline{P}'_2.$
$L'_{20},\ P'_{12},\ 79k \sim K^2_{14},\ 8P_4,\ Q_2,\ 12P_1;\ \Gamma_{64},\ 8P_{20},\ Q_{16},\ 12P_6.$
Type 7.
$C'_1 \sim C_5,\ \underline{P}_3,\ 12P_1.$
$C_1 \sim C'_5,\ \overline{P}'_4.$
$L'_{12},\ 40k \sim K^2_{12},\ P_8,\ 12P_2;\ \Gamma_{36},\ P_{20},\ 12P_8.$

Type 8.

$C'_1 \sim C_5,\ 2P_2,\ 13P_1.$
$C_1 \sim C'_5,\ 6\bar{P}'_2.$
$L'_{14},\ 56k \sim K^2_{12},\ 2P_5,\ 13P_2;\ \Gamma_{46},\ 2P_{18},\ 13P_{10}.$

Type 9.

$C'_1 \sim C_6,\ 2P_3,\ 14P_1.$
$C_1 \sim C'_6,\ 10\bar{P}'_2.$
$L'_{14},\ 57k \sim K^2_{15},\ 2P'_8,\ 14P_2;\ \Gamma_{54},\ 2P_{26},\ 14P_{10}.$

Type 10.

$C'_1 \sim C_7,\ P_3,\ 6P_2,\ 12P_1.$
$C_1 \sim C'_7,\ 15\bar{P}'_2.$
$L'_{18},\ 100k \sim K^2_{18},\ P_8,\ 6P_5,\ 12P_2;\ \Gamma_{90},\ P_{38},\ 6P_{26},\ 12P_{14}.$

8. Any two nets of curves having four variable intersections can be reduced by a series of quadratic transformations to a pair of nets of one of the ten types given in section 3. This method is the same as that used by Bertini* in showing that all pairs of nets intersecting in two variable points are birationally equivalent to a pair of nets of one of the three types of plane involutions of order two.

As an illustration of this method, consider a net of cubics with $6P_1$ and a net of quartics with $2P_2$, $4P_1$ at the basis points of the cubics. Choosing the $2P_2$ and any P_1 as the vertices of the triangle of inversion, the quartic net corresponds to a net of cubics with $5P_1$. The net of cubics with $6P_1$ is invariant, so the above system reduces to two nets of cubics with five common basis points, which is type 3.

A net of quintics with $16P_1$ at the basis points of a quartic pencil reduces to type 2 because the four remaining intersections of the quintic and a quartic of the pencil are collinear.

Two nets of quartics with 12 common basis points reduce to type 4 since the lines and points of (x') are in (1, 1) correspondence.

For plane involutions above the fourth order the number of types increases very rapidly. There is no relation between the order of the involution and the number of types except that one increases with the other.

9. Types 1, 2 and 5 are capable of generalization for any n. The results for type 1 will be given. In the defining equations written as in type 1 the φ_i now represent curves of order $m \geqq n$ with P of multiplicity $m - n$.

$C'_1 \sim C_{m+1},\ P_{m-n+1},\ n(2m - n + 1)P_1,$ genus $(n - 1)(m - n/2).$
$C_1 \sim C'_{m+1},\ P'_m.$
$L'_{(n-1)(2m-n+2)},\ P'_{(n-1)(2m-n)},\ 3(n - 2)(2m - n + 1)$ cusps,

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* E. Bertini, loc. cit.

$2(n-2)(n-3)(2m-n+1)$ nodes, genus $(2m-n)(2n-3)$.

K_{2m}, $P_{2(m-n+1)}$, $n(2m-n+1)P_1$, genus $(2m-n)(2n-3)$.
$\Gamma_{(n-2)(4m-n+1)}$, $P_{(n-2)(4m-3n+3)}$, $n(2m-n+1)P_{2(n-2)}$,

$3(n-2)(n-3)(2m-n+1)$ cusps, genus

$$(n-2)[(2m-n+1)(2n^2-10n+17)-4m]+1.$$

For $n > 4$ nothing new presents itself for the most general form of the defining equations since a general curve does not have tangents of higher multiplicity than two. So for any integer n in general there can be no more than two or three coincident images or a pair of coincidences. But the C_m of the defining equations may be so chosen that it will have any number of consecutive points or groups of consecutive points all collinear up to n and in these cases points of (x') lying at singular points of L' can be found whose coördinates cause this multiple tangent to pass through P and as many of whose images will coincide and in the same manner as this tangent has consecutive points of C_m. Similar conclusions apply to all types of plane involutions of order n.

10. For $m = n$ the $(1, n)$ Jonquières transformation (type 1 for $n = 4$) can be represented geometrically by means of a surface S of order $n+1$ with a simple point P_1 and an n-fold point P_n through which pass $n(n+1)$ lines of the surface.

Choose any two planes (x) and (x') as the simple and n-fold planes respectively. The line joining any point P'_i of (x') to P_1 cuts S in n points which project from P_n on (x) into n collinear points, the images in (x) of P'_i. The intersections of the line joining P_1 to P_n with (x) and (x') give the basis points P and P' respectively. Since the n intersections of P'_iP_1 with S lie in a plane determined by P_1P_n and P'_iP_1, the n image points of (x) are collinear with P.

Likewise to a point P_i of (x) corresponds one point in (x') found by projecting the single intersection of P_nP_i with S from P_1 on the plane (x').

The intersections of the $n(n+1)$ lines of S through P_n with (x) determine the $n(n+1)$ simple basis points of (x). The projection of these lines from P_1 on (x') gives the $n(n+1)$ basis lines of (x') all passing through P'.

Choose a line C'_1 of (x') not through P'. A plane through C'_1 and P_1 cuts S in a curve of order $n+1$ with a simple point at P and $n(n+1)$ fixed points on the lines of S. The projection of this curve on (x) from P_n gives C_{n+1}, the image of C'_1 with a simple point at P and $n(n+1)$ simple basis points. To two lines of (x') intersecting in P'_i correspond two image curves of (x) intersecting in the n images of P'_i all collinear with P.

When C'_1 passes through P' the plane determined by C'_1 and P_1 passes through P_n and the projection of the curve common to that plane and S from P_n on (x) is a line through P cut out by that plane. Thus the pencil of planes on P_1P_n cuts out the corresponding rays of the pencils on P and P' of (x) and (x') respectively. The $n(n + 1)$ fixed planes of the pencil determined by the $n(n + 1)$ lines of S cut out the $n(n + 1)$ basis lines of (x') and the lines of (x) through P and the respective P_1. Only proper images appear in this construction.

A plane through a line C_1 of (x) and P_n cuts S in a curve of order $n + 1$ with an n-fold point at P_n which projects from P_1 into a C'_{n+1} of (x') with P' of multiplicity n.

The tangent cone to S from P_1 is of order $(n - 1)(n - 2)$. From P_1 can be drawn a certain number of inflectional tangents and double tangents which are respectively cuspidal and double edges of the tangent cone. The intersection of this tangent cone with (x') gives the branch-point curve L' whose cusps and nodes are occasioned by the cuspidal and double edges of the cone. The basis lines of (x') are seen to be tangents to L'.

The projection from P_n on (x) of the contour curve common to S and the tangent cone gives the coincidence curve K. Each element of the tangent cone cuts S in $n - 2$ other points besides at the ordinary contacts. The projection on (x) from P_n of these residual intersections is the curve Γ. An inflectional contact projects from P_n into three consecutive points, two on K and one on Γ, so that K and Γ touch at that point and their common tangent passes through P since it is the projection of the inflectional tangent to S. The projection of the $n - 3$ residual intersections of the cuspidal edge of the tangent cone with S gives $n - 3$ cusps of Γ all lying on the common tangent to K and Γ. In the projection of a double contact from P_n each contact goes into a point of K and two consecutive points of Γ so that the projection of the bitangent to S is a bitangent to Γ through P. It is not tangent to K.

Each of the lines through P and a simple basis point of (x) is tangent to K at that simple basis point and has $n - 2$ contacts with Γ elsewhere.

For $n = 1$ the preceding becomes a three-dimensional construction for the ordinary quadratic transformation.

WELLS COLLEGE,
April, 1921.